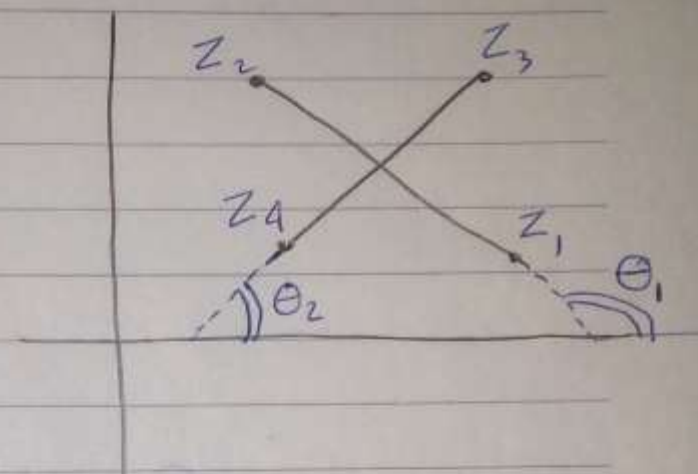
Proofs of Complex

1. Prove that if the line Joining the Points z_1, z_2 and z_3, z_4 are Perpendicular then:-

$$Re\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$$



Sol

$$\arg(z_1 - z_2) = \theta_1$$

$$\arg(z_3 - z_4) = \theta_2$$

$$\theta_1 = \theta_2 + \frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\arg(z_1 - z_2) - \arg(z_3 - z_4) = \frac{\pi}{2}$$

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \frac{\pi}{2}$$

$\therefore \frac{z_1 - z_2}{z_3 - z_4}$ تقع على محور y

$$\boxed{\therefore Re\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0}$$

$\boxed{2}$ use $z^n - 1 = 0$; $n = 2, 3, \ldots$ to show that

a) $\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = -1$

b) $\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0$

$\boxed{\text{Sol}}$

$z^n - 1 = 0 \Rightarrow z = (1)^{\frac{1}{n}} \Rightarrow x = 1, y = 0, r = 1, \theta = 0$

$z = r e^{i\left(\frac{\theta + 2k\pi}{n}\right)} = e^{\frac{2k\pi}{n} i}$

The roots $z_k = e^{\frac{2k\pi}{n} i}$; $k = 0, 1, \ldots$; $n = \ldots$

ـــ مجموع الجذور = صفر (معامل $z^{n-1} = 0$)

$z_0 + z_1 + z_2 + \cdots + z_{n-1} = 0$

$e^0 + e^{\frac{2\pi i}{n}} + e^{\frac{4\pi}{n} i} + \cdots + e^{\frac{2(n-1)\pi}{n} i} = 0$

$1 + \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right] + \left[\cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right)\right]$

$+ \cdots + \left[\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right] = 0$

Sub.
Date

ـــ نساوى الحقيق بالحقيقى :-

$$1+\cos\left(\frac{2\pi}{n}\right)+\cos\left(\frac{4\pi}{n}\right)+\cdots+\cos\frac{2(n-1)}{n}\pi=0$$

$$\cos\left(\frac{2\pi}{n}\right)+\cos\left(\frac{4\pi}{n}\right)+\cdots+\cos\frac{2(n-1)}{n}\pi=-1 \quad \# (a)$$

ـــ نساوى التخيلى بالتخيلى

$$\sin\left(\frac{2\pi}{n}\right)+\sin\left(\frac{4\pi}{n}\right)+\cdots+\sin\frac{2(n-1)}{n}\pi=0 \quad \# (b)$$

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[3] show that ~~E(z)~~

ـــ مر السكشن

$$1+\cos(\theta)+\cos(2\theta)+\cdots\cos(n\theta)=\frac{1}{2}+\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

Solution

$$e^{i\theta}=\cos\theta+i\sin\theta \; ; \; Re\left[e^{i\theta}\right]=\cos\theta$$

$$L.H.S=1+\cos\theta+\cos 2\theta+\cdots+\cos(n\theta)$$

$$=Re\left[1+e^{i\theta}+e^{i2\theta}+\cdots+e^{in\theta}\right]$$

Sub.
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مجموع المتوالية الهندسية :-

$$a + ar + ar^2 \dots = a \frac{1-r^{n+1}}{1-r}$$

here: $a = 1$, $r = e^{i\theta}$

$$L.H.S = Re\left[\frac{1 - e^{i\theta(n+1)}}{1 - e^{i\theta}}\right] \quad \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} \ * \text{بالضرب}$$

$$= Re\left[\frac{e^{-i\frac{\theta}{2}} - e^{i(n+1)\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}\right]$$

$$= Re\left[\frac{\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}) - \cos\left[(n+1)\frac{\theta}{2}\right] - i\sin\left[(n+1)\frac{\theta}{2}\right]}{\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})}\right]$$

$$= \frac{-\sin\frac{\theta}{2} - \sin(n+1)\frac{\theta}{2}}{-2\sin\frac{\theta}{2}}$$

$$L.H.S = \frac{1}{2} + \frac{\sin(n+1)\theta}{2\sin\frac{\theta}{2}} = R.H.S \ \#$$

Sub.
Date

4 show that

صبر المسكشر

$$|Z_1+Z_2|^2+|Z_1-Z_2|^2=2|Z_1|^2+2|Z_2|^2$$

Sol

$$L.H.S=|Z_1+Z_2|^2+|Z_1-Z_2|^2$$

$$=(Z_1+Z_2)(\overline{Z_1+Z_2})+(Z_1-Z_2)(\overline{Z_1-Z_2})$$

$$=Z_1\bar{Z}_1+\cancel{Z_1\bar{Z}_2}+\cancel{Z_2\bar{Z}_1}+Z_2\bar{Z}_2+Z_1\bar{Z}_1-\cancel{Z_1\bar{Z}_2}$$

$$-\cancel{Z_2\bar{Z}_1}+Z_2\bar{Z}_2$$

$$L.H.S=|Z_1|^2+|Z_2|^2+|Z_1|^2+|Z_2|^2$$

$$L.H.S=2|Z_1|^2+2|Z_2|^2=R.H.S$$

//

Sub.
Date

5 show that $f(z) = \bar{z} = x - iy$ is not differentiable at $z_0 = 0$

→ solution ←

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}, \quad f(z) = \bar{z}, \quad f(0) = 0$$

$$f(0+\Delta z) = f(\Delta z) = \overline{\Delta z} = \overline{\Delta x + i\Delta y} = \Delta x - i\Delta y$$

$$f'(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

→ since $\lim \lim \neq \lim \lim$

∴ the limit doesn't exist

∴ not diff.

6 usei C-R equations to show that

$$\frac{\partial z^n}{\partial z} = nz^{n-1}$$

مطلوب ان نوجد قانون المشتقة الأولى
(C-R) ونوضح انها بعد الاختصار
تعطى nz^{n-1}

$\boxed{\text{Sol}}$

$$f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= \underbrace{r^n \cos(n\theta)}_{u} + \underbrace{r^n \sin(n\theta)}_{v}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= e^{-i\theta}\left[nr^{n-1}\cos(n\theta) + inr^{n-1}\sin(n\theta)\right]$$

$$= nr^{n-1}e^{-i\theta}\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$e^{i*} = \cos * + i\sin *$$

$$f'(z) = nr^{n-1}e^{-i\theta}\left[e^{in\theta}\right]$$

$$f'(z) = n r^{n-1} e^{i(n-1)\theta} = n(re^{i\theta})^{n-1}$$

$$\therefore f'(z) = n z^{n-1}$$

[7] Prove that $\frac{d}{dz}(z^2 \bar{z})$ doesn't exist any where.

السكشن

[Sol]

$$f(z) = z^2 \bar{z} = (x+iy)^2 (x-iy)$$

$$= (x^2 + i2xy - y^2)(x-iy)$$

$$= x^3 + i2x^2y - xy^2 - ix^2y + 2xy^2 + iy^3$$

$$f(z) = \underbrace{(x^3 + xy^2)}_{u} + i\underbrace{(y^3 + x^2y)}_{v}$$

$u_x = 3x^2 + y^2$ $\quad$ $u_y = 2xy$

$v_x = 2xy$ $\quad$ $v_y = 3y^2 + x^2$

$u_x \neq v_y$, $v_x \neq -v_y$

not analytic $\Rightarrow$ not diff

Sub.
Date

$\boxed{8}$ show that if $f(z) = u+iv$ is analytic

then:

$$\nabla^2 |f(z)|^2 = 4\left|\frac{df}{dz}\right|^2$$

$|f(z)| \to u^2+v^2$

$\nabla^2 \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Sol

$$\frac{\partial}{\partial x}(u^2+v^2) = 2uu_x + 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(u^2+v^2) = 2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2 \to ①$$

$$\frac{\partial^2}{\partial y^2}(u^2+v^2) = 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2V_y^2 \to ②$$

① + ②

$$= 2u(u_{xx}+u_{yy}) + 2(u_x^2+V_x^2)$$

$$+ 2u(V_{xx}+V_{yy}) + 2(u_y^2+V_y^2)$$

$\because u_{xx}+u_{yy}=0$ ، $V_{xx}+V_{yy}=0$

$u_x^2 + V_x^2 = f'(z)$

$u_y^2 + V_y^2 = f'(z)$

$① + ② = 4\left|\frac{df}{dz}\right|^2$

$\boxed{9}$

a) show that if $f(z)=u(x,y)+iV(x,y)$ is analytic then $u(x,y)$ and $V(x,y)$ are harmonic

b) ~~show that f~~ → solution ⟵

* analytic function:-

$u_x = V_y \rightarrow (1)$ ، $u_y = -V_x \rightarrow (2)$

* harmonic function:-

$u_{xx}+u_{yy}=0$

بتفاضل رقم (1) بالنسبة لـ x

$\therefore u_{xx} = V$

Sub.
Date

ثم بتفاضل رقم (٢) بالنسبة لـ y

$$U_{yy} = -V_{xy} \longrightarrow (b)$$

$\boxed{a+b}$

$U_{xx} + U_{yy} = 0$ ∴ is harmonic #

10 show that if $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic then $u(r,\theta)$ and $v(r,\theta)$ are harmonic

solution

→ analytic function :-

$$U_r = \frac{1}{r} V_\theta \quad , \quad V_r = \frac{-1}{r} U_\theta$$

→ Harmonic function.

$$r^2 U_{rr} + r U_r + U_{\theta\theta} = 0$$

$\Rightarrow r U_r = V_\theta$ تفاضل بالنسبة لـ r

$$r U_{rr} + U_r = V_{\theta r} \longrightarrow (1)$$

$\Rightarrow r V_r = -U_\theta$ تفاضل بالنسبة لـ θ

$$r V_{\theta r} = -U_{\theta\theta} \longrightarrow (2)$$

Sub.
Date

$V_{r\theta} = V_{\theta r}$ with (1), (2)

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \quad \#$$

$\boxed{11}$ show that $\ln \frac{x+iy}{x-iy} = 2i \tan^{-1} \frac{y}{x}$

$\boxed{\text{sol}}$

$z = x+iy = re^{i\theta}$, $\bar{z} = x-iy = re^{-i\theta}$

$r = \sqrt{x^2+y^2}$ $\qquad \theta = \tan^{-1} \frac{y}{x}$

$$\ln \frac{x+iy}{x-iy} = \ln \frac{re^{i\theta}}{re^{-i\theta}}$$

$$= \ln e^{2i\theta} = 2i\theta$$

$$= 2i \tan^{-1} \frac{y}{x} \quad \#$$

Sub.
Date

12 show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Solution

$\cos z = \cos(x+iy)$

$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$

$= \cos(x)\cdot\cosh(y) - \sin(x)\sinh(y)$

$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$\because \sin^2 x = 1 - \cos^2 x$, $\cosh^2 y = 1 + \sinh^2 y$

$|\cos z|^2 = \cos^2(x)(1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$ #

$\boxed{13}$ show that

$$\cosh^{-1} z = \ln\left(z \pm \sqrt{z^2 - 1}\right)$$

$\boxed{\text{sol}}$

assume:

$$\cosh^{-1} z = w \implies z = \cosh w$$

$$z = \frac{e^{w} + e^{-w}}{2} \implies e^{w} + e^{-w} = 2z$$

بالضرب ب × e^{w}

$$\left(e^{w}\right)^2 - 2z e^{w} + 1 = 0$$

$\underline{\underline{\text{roots}}} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$ e^{w}

$$w = \ln\left(z \pm \sqrt{z^2 - 1}\right)$$

$$\therefore \cosh^{-1} z = \ln\left(z \pm \sqrt{z^2 - 1}\right)$$

14 show that:-

$$\frac{d}{dz}\left(Ln(z)\right) = \frac{1}{z}$$

Sol

$$f(z) = Ln\, z = \underbrace{Ln(r)}_{r} + \underbrace{i\theta}_{v}$$

$u_r = \frac{1}{r}$ $\quad$ $u_\theta = 0$

$\frac{-1}{r}$ $\quad$ $\frac{1}{r}$

$v_r = 0$ $\quad$ $v_\theta = 1$

f_n is analytic

$$f'(z) = (u_r + i v_r)\, e^{-i\theta} = \frac{1}{r} e^{-i\theta}$$

$$= \frac{1}{r e^{i\theta}} \quad \text{where: } r e^{i\theta} = z$$

$$\therefore f'(z) = \frac{1}{z} \quad \#$$

Sub.
Date

$\boxed{15}$ show that $(1+i)^i = e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)} \cdot e^{\frac{i}{2} \ln(2)}$

Solution

$$L.H.S = (1+i)^i = e^{\ln(1+i)^i} = e^{i \ln(1+i)}$$

$$= \cos(\ln(1+i)) + i \sin(\ln(1+i))$$

$\ln(1+i)$ $\quad r = \sqrt{2}$ $\quad \theta = \frac{\pi}{4}$

$$\ln(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} \pm 2n\pi\right)$$

$$L.H.S = e^{i\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} \pm 2n\pi\right)\right]}$$

$$= e^{i \ln(\sqrt{2})} \cdot e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)}$$

$\sqrt{2} = (2)^{\frac{1}{2}}$

$$= e^{\frac{i}{2} \ln(2)} \cdot e^{-\left(\frac{\pi}{4} \pm 2n\pi\right)}$$

$= R.H.S$ #

Sub.
Date

16 show that
$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Solution

$$\sin(z) = \sin(x+iy)$$

$$= \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$= \sin(x)\cdot\cosh(y) + \cos(x)\cdot\sinh(y)$$

$$|\sin z|^2 = \sin^2 x\cdot\cosh^2 y + \cos^2 x\,\sinh^2 y$$

$$= \sin^2 x\left[1+\sinh^2 y\right] + (1-\sin^2 x)\sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad //$$

17 show that

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right)$$

L.H.S. $= \sinh^{-1} z = w$

$$z = \sinh z = \frac{e^{w} - e^{-w}}{2} \Rightarrow e^{w} - e^{-w} = 2z$$

بالضرب * e^{w}

$$e^{2w} - 2ze^{w} - 1 = 0$$

$$e^{w} = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}$$

$$w = \ln\left(z \pm \sqrt{z^2+1}\right)$$

where $w = \sinh^{-1} z$ #

18 **Prove that :-**

$$\int_C (z-z_0)^m dz = \begin{cases} 2\pi i & m=-1 \\ 0 & \text{otherwise.} \end{cases}$$

where: C is $|z-z_0| = a$

$\boxed{\text{sol}}$

$m \neq -1$

$|z-z_0| = a \Rightarrow z - z_0 = a e^{i\theta}$

$dz = i a e^{i\theta} d\theta \qquad 0 \leqslant \theta \leqslant 2\pi$

$$I = \int_0^{2\pi} (a e^{i\theta})^m \, i a e^{i\theta} \, d\theta$$

$$= i a^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} \, d\theta$$

$$= i a^{m+1} \left. \frac{e^{i(m+1)\theta}}{i(m+1)} \right|_0^{2\pi}$$

Sub.
Date

$$I = \frac{i a^{m+1}}{i(m+1)} \left[e^{2\pi(m+1)i} - 1 \right]$$

$\because \sin(n\pi) = 0 \quad , \cos(n\pi) = (-1)^n$

$$e^{2(m+1)\pi i} = \cos 2(m+1)\pi + i \sin 2(m+1)\pi$$

$$= (-1)^{2(m+1)} = \underline{1}$$

$\therefore \quad I = 0$

<u>at $m = -1$</u>

$$I = \int_c \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i a e^{i\theta} \, d\theta}{a e^{i\theta}} = 2\pi i$$

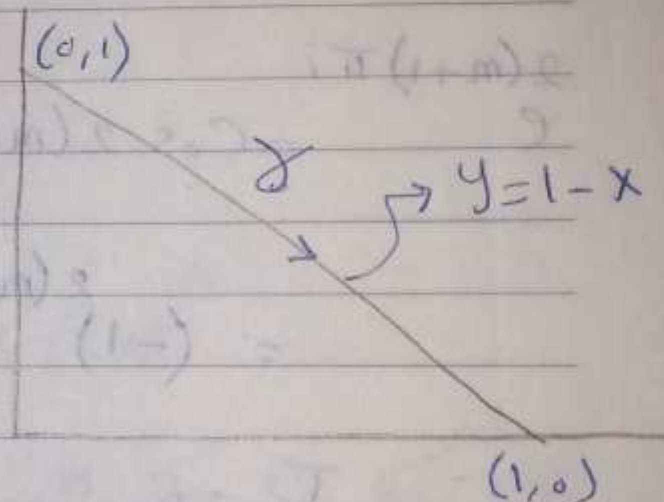
Sub.
Date

[19] Let γ denote the line segment from $z=i$ to $z=1$.

Show that $\left| \int_\gamma \frac{dz}{z^4} \right| \leqslant 4\sqrt{2}$

Sol

$$\left| \int_\gamma \frac{dz}{z^4} \right| \leqslant ML$$



$$L(\gamma) = \sqrt{2}$$

$$|f(z)| = \left| \frac{1}{z^4} \right| = \frac{1}{|z^4|}$$

$$|z^4| = \left(\sqrt{x^2+y^2}\right)^4 = \left(x^2+y^2\right)^2$$

من معادلة المنحنى

$$= \left[x^2 + (1-x)^2\right]^2 = \left(x^2 + 1 - 2x + x^2\right)^2$$

$$|z|^4 = 4\left[x^2 - x + \frac{1}{2}\right]^2$$

إكمال مربع

(جذر الأول الإشارة $\frac{1}{2}$ معامل الثانى$)^2$ - ($\frac{1}{2}$ معامل الثانى$)^2$ + الباقى

$$|z|^4 = 4\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2}\right]^2$$

$$= 4\left[\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\right]^2$$

$$|f(z)| = \frac{1}{4\left[\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\right]^2} \quad 0 \leq x \leq 1$$

أعلى قيمة للمقدار عند $x = \frac{1}{2}$ ←

$$|f(z)| \leq 4$$

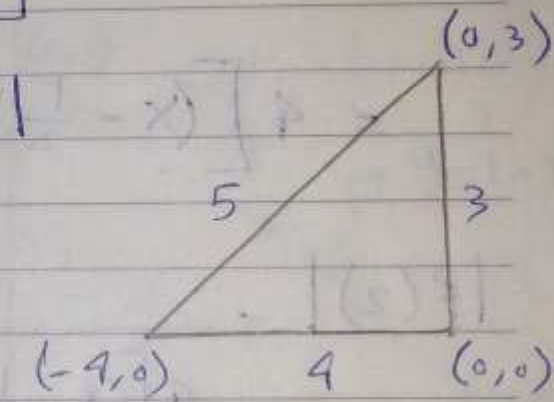
$$\left|\int_\gamma f(z)\,dz\right| \leq ML$$

$$\leq 4\sqrt{2}$$

Sub.
Date

(20) show that $\left|\int_\gamma (e^z - \bar{z})\,dz\right| \leq 60$; where γ denote the boundary of triangle with vertices $z=0$, $z=-4$ and $z=3i$

solution



$f(z) = |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$

$\leq |e^{x+iy}| + |x-iy|$

$\leq |e^x \cdot e^{iy}| + \sqrt{x^2+y^2}$

$\leq e^x |\cos(y) + i\sin(y)| + \sqrt{x^2+y^2}$

$\leq e^x \sqrt{\cos^2 y + \sin^2 y} + \sqrt{x^2+y^2}$

$\leq e^x + \sqrt{x^2+y^2}$

at $(0,0) \longrightarrow |e^z - \bar{z}| \leq 1$

at $(0,3) \longrightarrow |e^z - \bar{z}| \leq 4$

at $(-4,0) \longrightarrow |e^z - \bar{z}| \leq 4{,}02 \longrightarrow M$

Sub.
Date

$$\int_{\gamma} |f(z)|\,dz \leq 6 \quad \#$$

(21) show that if $f(z)$ is analytic on simple closed curve then $\oint_c f(z)\,dz = 0$

~~Note that~~

solution

Note that

$$\oint P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\oint_c f(z)\,dz = \oint_c (u + iv)(dx + i\,dy)$$

$$= \oint_c (u\,dx - v\,dy) + i \oint_c (v\,dx + u\,dy)$$

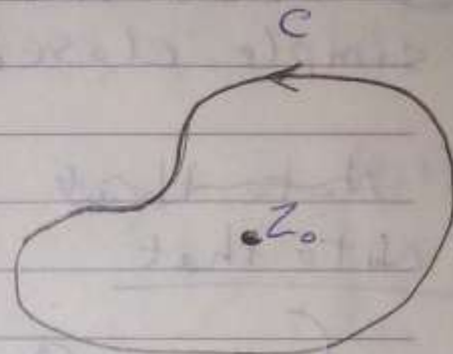
$$= \iint_D \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

sinze $f(z)$ is analytic $\rightarrow$ $u_x = v_y$, $u_y = -v_x$

$$\oint_c f(z)\,dz = 0 \quad \#$$

22 show that if $f(z)$ is analytic on s.cc and z_0 inside C then

$$\oint \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$$

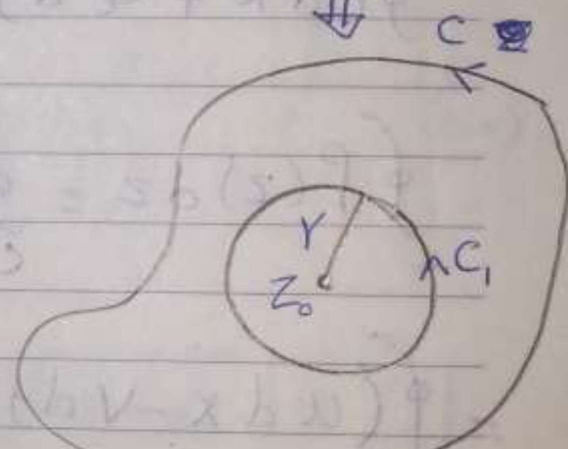


Sol

نفرض دائرة مركزها Z_0 ونصف قطرها r

$|z-z_0| = r$

$$\oint = \oint_{C_1}$$



$$I = \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0}\, dz$$

$$= \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta$$

as: $r \longrightarrow 0$

$$I = i\, f(z_0) \int_0^{2\pi} d\theta = 2\pi i\, f(z_0) \quad \#$$

23 show that if $f(z)$ is analytic and bounded then $f(z)$ must be constant. use Cauchy integral form.

Sol

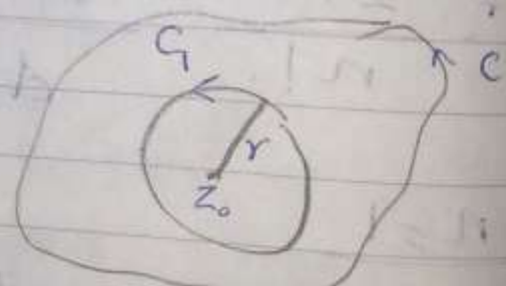
C.I

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\,\frac{d^n f(z)}{dz^n}\Big|_{z=z_0}$$

$|f(z)| \leq M$ ـ الدالة تكون محدودة اذا كانت

ـ نفرض دائرة مركزها z_0 ونصف قطرها (r) داخل c.

$$\oint = \oint_{C_1}$$



Note $\left|\int f\right| \leq \int |f|$.

$$\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \oint \frac{|f(z)|}{|z-z_0|^{n+1}}\,dz$$

$$|z-z_0| = r$$

Sub.
Date

$$\left|\oint_{c_1}\right| \leq \frac{M}{r^{n+1}} \oint_{c} \underbrace{|dz|}$$

وحدة طول (جزء من الدائرة وتجميعها يعطي محيط الدائرة بالكامل)

① $$\left|\oint_{c_1}\right| \leq \frac{M}{r^{n+1}}(2\pi r)$$

$$\leq \frac{M}{r^{n}}(2\pi)$$

$$\therefore \frac{2\pi i}{n!} \left.\frac{d^n f(z)}{dz^n}\right|_{z=z_0} \leq \frac{2\pi M}{r^n} \; ; |i| = 1$$

$\underline{n=1}$

$$|f'(z_0)| \leq \frac{M}{r}$$

as $r \to \infty \Rightarrow |f'(z_0)| \leq 0$ لا يوجد مقياس سالب

$$f'(z_0) = 0 \longrightarrow f(z_0) = c \text{ for all } z.$$

$f(z) = \text{constant}$ #

Sub.
Date

24 use Laurent's series to show that $\oint_c f(z)\,dz = 2\pi i\, a_{-1}$ wher $f(z)$ is analytic on the region $r < |z-z_0| < R$

Sol

use

$$\oint_c (z-z_0)^m \, dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \end{cases} \rightarrow (1)$$

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z-z_0)^n$$

$$f(z) = \cdots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

بالضرب × $(z-z_0)^m$

$$f(z) * (z-z_0)^m = \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} \cdots\cdots$$

بالتكامل على المنحنى وإستخدام الصورة (1)

at $m=0 \Rightarrow \oint f(z)\,dz = a_{-1}(2\pi i)$